Midterm Exam, MAT354

October 17, 2023

This is the 2023 midterm, posted for students to practice.
Four problems, 10 points each; total 40 points.

Problem 1 *10 points*

Let $z_0 \neq 0 \in \mathbb{C}$. Find the farthest point w of the Complex plane from z_0 in spherical metric, i.e. a point $w \in \mathbb{C}$ such that if $z \neq w$ then

$$\mathrm{dist}_{\mathrm{spherical}}(z_0,\, w) > \mathrm{dist}_{\mathrm{spherical}}(z_0,\, z).$$

Does such a point w exist for $z_0 = 0$?

Problem 2 *10 points*

1. Let $(x_j)_{j=1}^n$ be a finite sequence of real numbers. Show that if $\operatorname{Im} z > 0$ and $a \in \mathbb{R}$, $a > 0$, then
$$\left|\prod_{j=1}^{n}(z - x_j - ai)\right| < \left|\prod_{j=1}^{n}(z - x_j + ai)\right|.$$
2. Let $P(z)$ be a polynomial of degree n with all the roots real and $a \in \mathbb{R}$. Show that all the roots of the polynomial $Q(z) := P(z - ai) + P(z + ai)$ are also real.

Problem 3 *10 points*

Assume that (a_n) is a sequence of complex numbers such that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$. Let $R > 0$. Prove that for any $\rho > \max(1,\, R)$ the series

$$\sum_{n=0}^{\infty} \frac{R^n a_n}{z^n + z^{-n}}$$

converges uniformly on the set $\{z \in \mathbb{C} : |z| \geq \rho\}$.

Problem 4 *10 points*

Prove that

$$\lim_{y\to\infty} \frac{1}{\cos(x + iy)} = 0$$

uniformly in x, i.e. prove that

$$\forall \varepsilon > 0\ \exists R \text{ such that if } |y| > R \text{ then } \left|\frac{1}{\cos(x + iy)}\right| < \varepsilon.$$